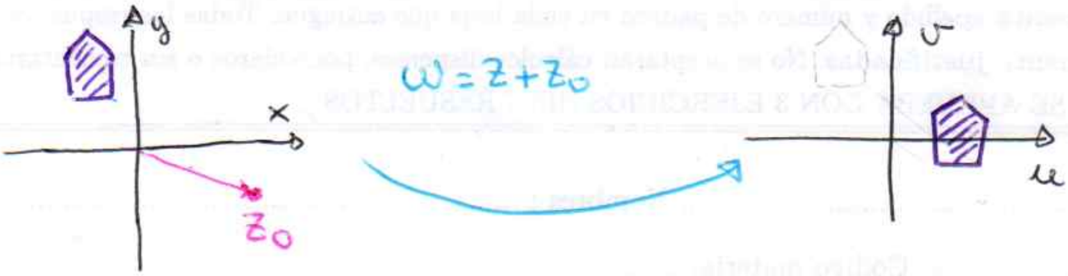
Transformaciones del plano

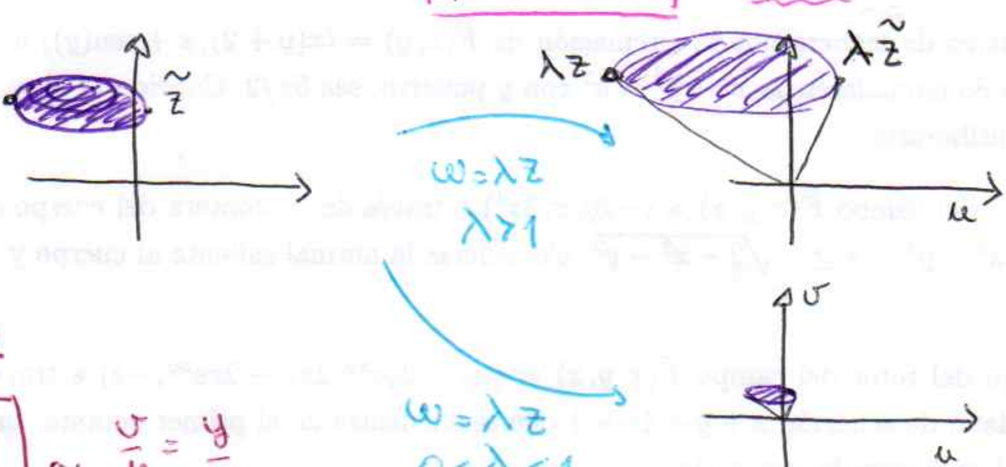
(A) Traslación

$$\boxed{f(z) = z + z_0}$$



(B) Cambio de escala

$$\boxed{f(z) = \lambda z}, \quad \lambda \in \mathbb{R}^+$$



$w = \lambda z$

$$\boxed{\begin{array}{l} u = \lambda x \\ v = \lambda y \end{array}} \sim \frac{v}{u} = \frac{y}{x}$$

(C) Rotación

$$\boxed{f(z) = e^{i\alpha} \cdot z} = e^{i\alpha} \cdot r e^{i\theta} = r e^{i(\theta + \alpha)}$$

$z = r e^{i\theta}$

(B) + (C): $\boxed{f(z) = a \cdot z}$ $\quad a \in \mathbb{C} \quad a = \lambda e^{i\alpha}$

↳ rotación + cambio de escala.

(D) Cuadrática $\boxed{f(z) = z^2}$

$w = z^2$

Real cartesiana? $\rightarrow$ $\boxed{\begin{array}{l} u = x^2 - y^2 \\ v = 2xy \end{array}}$

Rectas $x = a$ $(a \in \mathbb{R})$ $\longrightarrow$ $\left.\begin{array}{l} u = a^2 - y^2 \\ v = 2ay \end{array}\right\} \rightarrow$ $y = v/2a$ $(a \neq 0)$

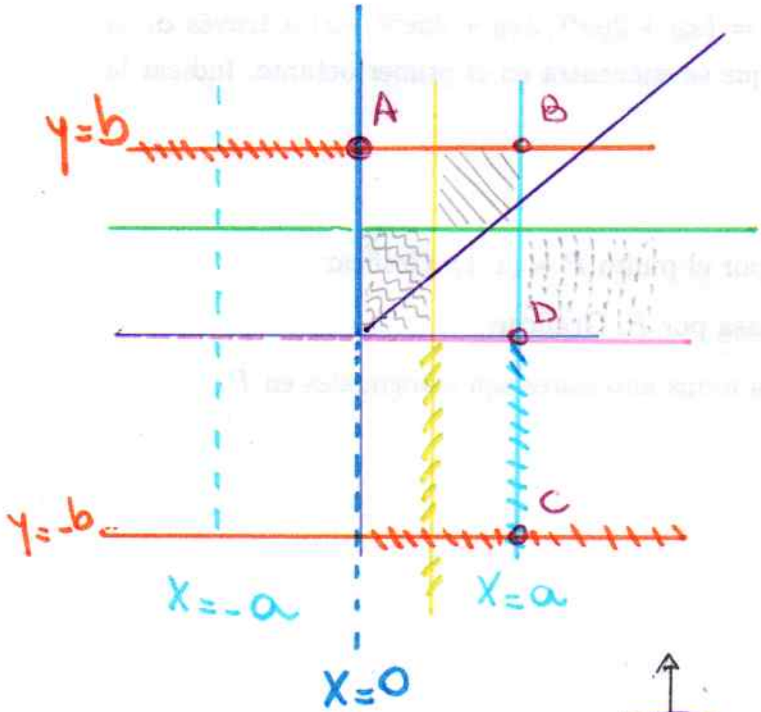
$u = a^2 - \dfrac{v^2}{4a^2}$ $\rightarrow$ parábola

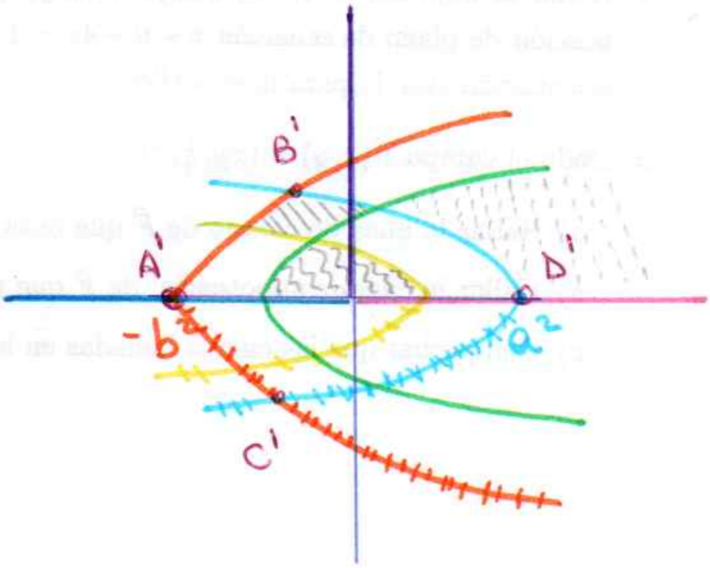
$\rightarrow x = 0$ $\longrightarrow$ $\left.\begin{array}{l} u = -y^2 \leq 0 \\ v = 0 \end{array}\right\} \rightarrow$ semieje real negativo

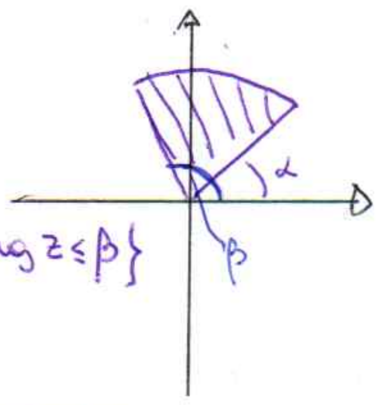
Rectas $y = b$ $(b \in \mathbb{R})$ $\longrightarrow$ $\left.\begin{array}{l} u = x^2 - b^2 \\ v = 2xb \end{array}\right\}$ $x = v/2b$ $(b \neq 0)$

$u = \dfrac{v^2}{4b^2} - b^2$ $\rightarrow$ parábola en plano (u, v)

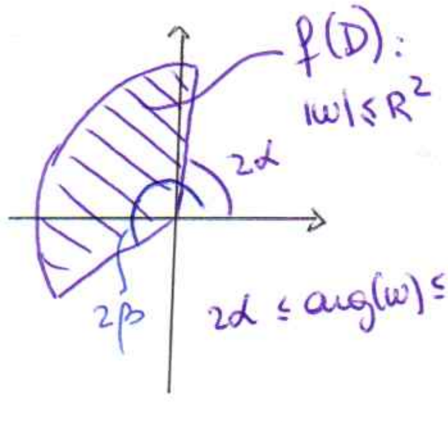
$\rightarrow y = 0$ $\longrightarrow$ $\left.\begin{array}{l} u = \cancel{x}^2 \geq 0 \\ v = 0 \end{array}\right\}$ semieje real positivo







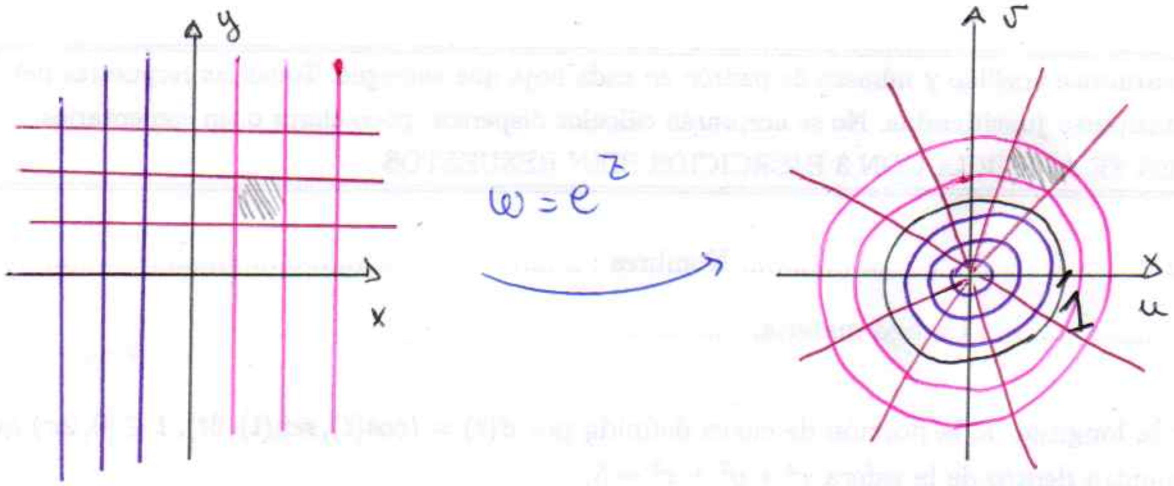
$D = \{z \in \mathbb{C} : |z| \leq R,\ \alpha \leq \arg z \leq \beta\}$



$2\alpha \leq \arg(w) \leq 2\beta$

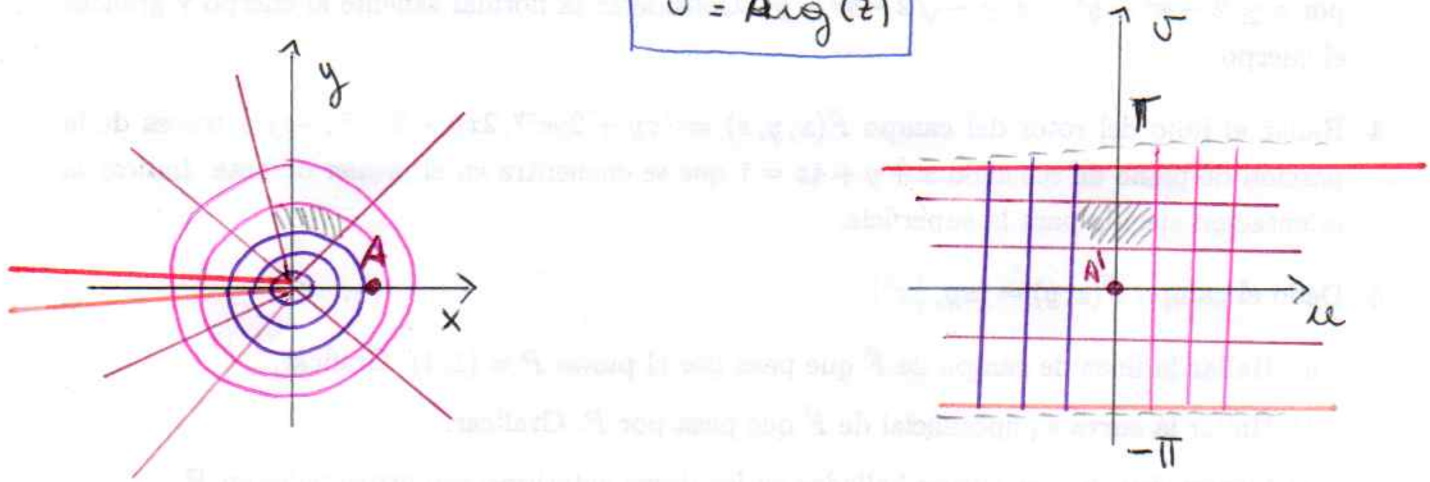
(E) Exponencial $\boxed{w = e^{z}}$

$$\boxed{\begin{array}{l} u = e^{x}\cos y \\ v = e^{x}\,\text{sen}\,y \end{array}}$$



(F) Logaritmo principal. $\boxed{w = \text{Log}\, z}$

$$\boxed{\begin{array}{l} u = \ln|z| \\ v = \text{Arg}(z) \end{array}}$$



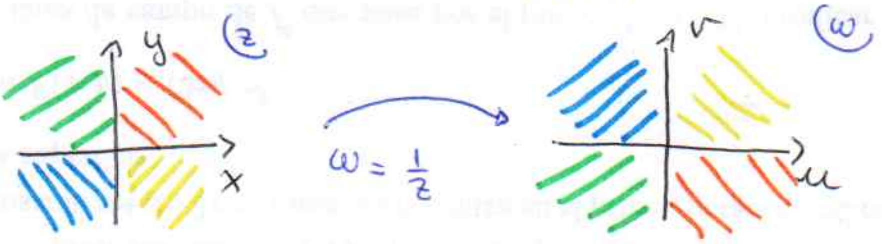
(6) Inversión

$$\boxed{w = \frac{1}{z}}$$

$$\boxed{\begin{aligned} u &= \frac{x}{x^2+y^2} \\ v &= \frac{-y}{x^2+y^2} \end{aligned}}$$

$x > 0 \iff u > 0 \rightsquigarrow$ semiplano derecho (en z) $\longleftrightarrow$ semiplano derecho (en w)

$y > 0 \iff v < 0 \rightsquigarrow$ semiplano superior en (z) $\longleftrightarrow$ semiplano inferior (en w)

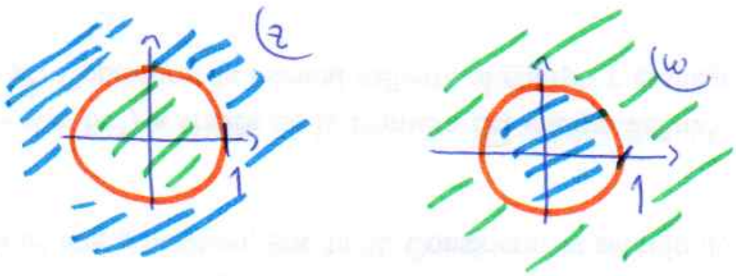


Como $|w| = \frac{1}{|z|}$, entonces:

Si $|z| > 1 \implies |w| < 1$

$|z| = 1 \implies |w| = 1$

$|z| < 1 \implies |w| > 1$



Red cartesiana

$x = a \neq 0$

$$\left.\begin{aligned} u &= \frac{a}{a^2+y^2} \\ v &= \frac{-y}{a^2+y^2} \end{aligned}\right\} \quad u^2+v^2 = \frac{a^2}{(a^2+y^2)^2} + \frac{y^2}{(a^2+y^2)^2} = \frac{1}{a^2+y^2} = \frac{u}{a}$$

$$u^2 - \frac{u}{a} + v^2 = 0$$

$$\left\{ \left(u - \frac{1}{2a}\right)^2 + v^2 = \frac{1}{4a^2} \right\} \quad \text{si } a \neq 0$$

$y = b \neq 0$

$$\left.\begin{aligned} u &= \frac{x}{x^2+b^2} \\ v &= \frac{-b}{x^2+b^2} \end{aligned}\right\} \quad u^2+v^2 = \frac{x^2}{(a^2+b^2)^2} + \frac{b^2}{(x^2+b^2)^2} = \frac{1}{x^2+b^2} = -\frac{v}{b}$$

$$u^2 + v^2 + \frac{v}{b} = 0 \iff \left\{ u^2 + \left(v + \frac{1}{2b}\right)^2 = \frac{1}{4b^2} \right\}$$

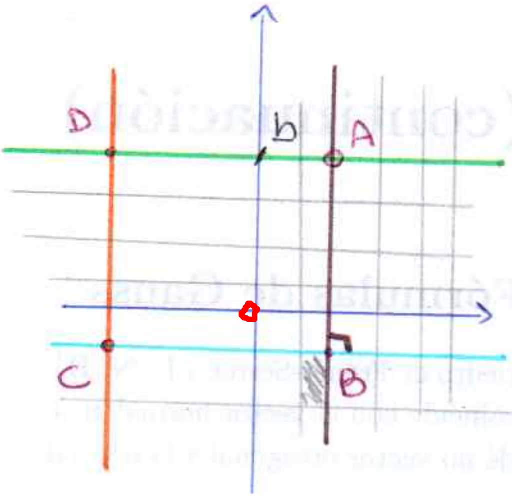
si $b \neq 0$

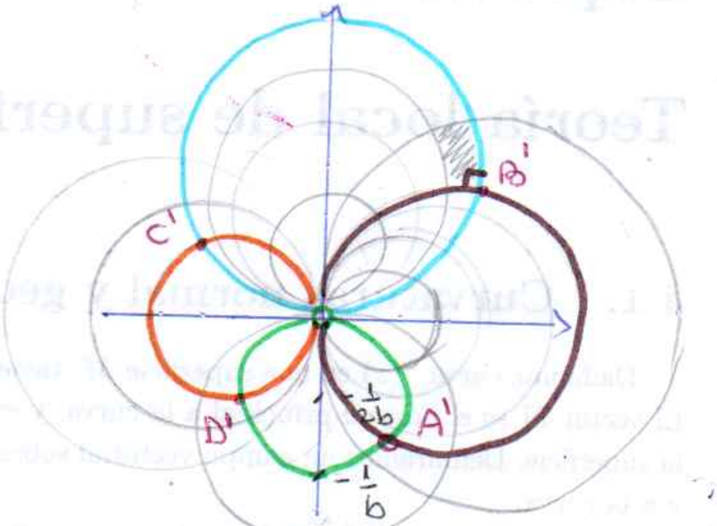
$x=0 \iff u=0$

$y=0 \iff v=0$

$$u = \frac{x}{x^2+y^2}$$

$$v = \frac{-y}{x^2+y^2}$$





En general: rectas y circunferencia en z $\iff$ rectas y circunferencia en w

$$\boxed{a(x^2+y^2) + bx + cy + d = 0}$$

Observe: $a=0 \iff$ recta $\quad d=0 \iff$ pasa por origen

$a \neq 0 \iff$ circunferencia

$\hookrightarrow$ $b\neq 0$ o $c \neq 0$: centro fuera del origen

$$x = \frac{u}{u^2+v^2} \qquad \left(\text{porque } w = \frac{1}{z} \iff z = \frac{1}{w}\right)$$

$$y = \frac{-v}{u^2+v^2}$$

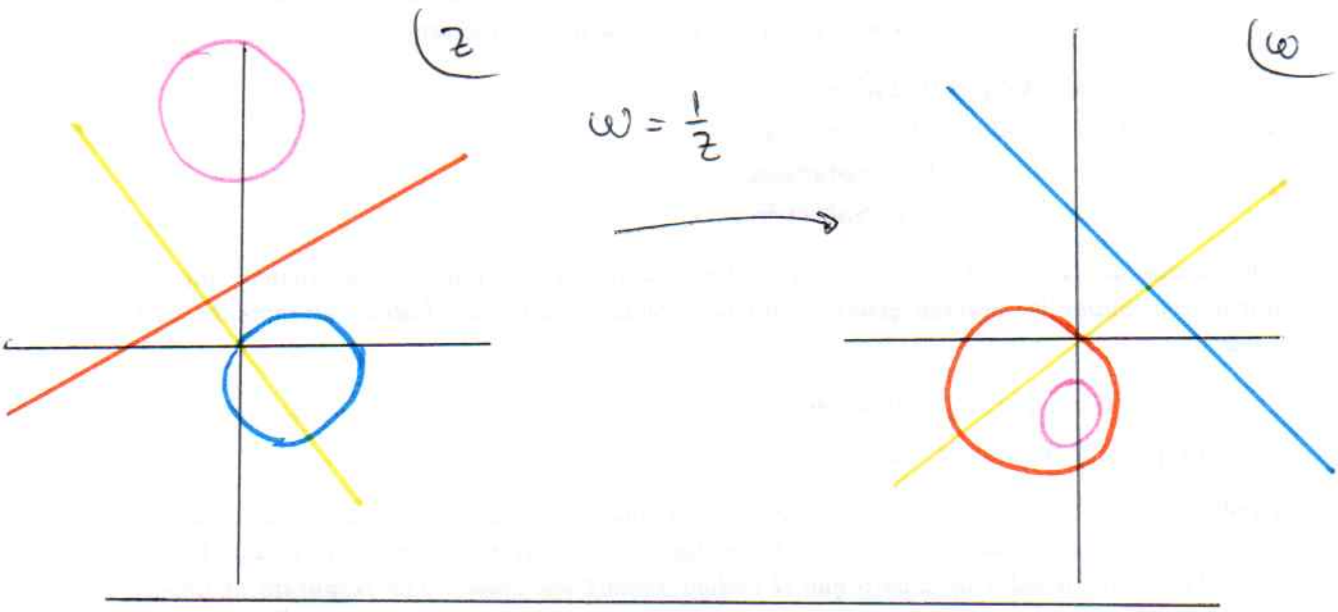
$$a\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + b\frac{u}{u^2+v^2} + c\frac{(-v)}{u^2+v^2} + d = 0$$

$$\boxed{a + bu - cv + d(u^2+v^2) = 0}$$

$$a(x^2+y^2) + bx + cy + d = 0$$

$$a + bu - cv + d(u^2+v^2) = 0$$

	$d=0$	$d\neq 0$
$a=0$	recta por el origen en recta por el origen	recta fuera del origen en circunferencia por origen
$a\neq 0$	circunferencia por origen recta fuera del origen	circunferencia fuera del origen circunferencia fuera del origen



(H) Transformación de Möbius - o racional lineal o homográfica

$$\boxed{w = \frac{az+b}{cz+d}} \qquad ad-bc \neq 0 \qquad a,b,c,d \in \mathbb{C}$$

Si $c \neq 0$: $w = \frac{a}{c} + \frac{(bc-ad)}{c} \cdot \frac{1}{cz+d}$ → composición de

$z_1 = cz+d$

$z_2 = 1/z_1$

$z_3 = w = \frac{a}{c} + \frac{(bc-ad)}{c} \cdot z_2$

Extendida a $\mathbb{C}^*$ (plano complejo ampliado)

si $c \neq 0$

$$w = \begin{cases} \frac{a}{c} + \frac{(bc-ad)}{c}\frac{1}{cz+d} & \text{si } z \neq -\frac{d}{c}, \neq \infty \\ \infty & \text{si } z = -\frac{d}{c} \\ a/c & \text{si } z = \infty \end{cases}$$

si $c = 0$

$$w = \begin{cases} \frac{a}{d} z + \frac{b}{d} & \text{si } z \neq \infty \\ \infty & \text{si } z = \infty \end{cases}$$

Propiedades

* la inversa de una homografía es una homografía
* la composición de homografías es una homografía
* una homografía transforma rectas y circunferencias en rectas y circunferencias